Exam Random Geometry and Topology B

7 November 2023

- You have from 08.30 until 10.30.
- It is not allowed to use phones, computers, books, notes or any other aids.
- Write each exercise on a separate sheet of paper, with your name and student number clearly legible on each sheet.

Consider the **one-sided contact process** $(\xi_t)_{t\geq 0}$ with parameter $\lambda > 0$ on the 2-dimensional lattice, that is, on the state space $S = \{0,1\}^{\mathbb{Z}^2}$. This is a version of the contact process in which infected individuals can only transmit the infection to their south and west neighbours.

More precisely, we start the process from an initial configuration of healthy (state 0) and infected (state 1) sites at $t = 0$. If a site $x \in \mathbb{Z}^2$ is infected, then it becomes healthy after an exponential time with parameter 1 independently of the state of the other sites. Furthermore, it transmits the infection after an exponential time with parameter λ (that is, with mean $1/\lambda$) to $x - (1,0)$ and (independently of that) transmits the infection after an exponential time with parameter λ to $x - (0,1)$.

We will abuse notation and identify a configuration $\xi \in S$ with the set of infected vertices in ξ.

Exercise 1 (10 pts)
Write down the generator of $(\xi_t)_{t\geq 0}$. Is the process monotone? Why/why not?

Exercise 2 (15 pts)
Describe a graphical representation of the process and define ξ_t, the state of the process at some time $t > 0$, in terms of this representation.

Exercise 3 (15 pts)
Denote by ξ_t^o the sate of the process at time t starting from a single infection at the origin o at time 0 and let the extinction time be

$$\tau := \inf\{t : \xi_t^o = \varnothing\}.$$

Define

$$\lambda_c := \inf\{\lambda : \mathbb{P}(\tau = \infty) > 0\}.$$

Prove that $0 < \lambda_c < \infty$.
(In the solution bounds for the critical parameter of other processes proven in the lectures or tutorials may be used without proof.)

Exercise 4 (15 pts)
Let $(\tilde{\xi}_t)_{t\geq 0}$ be the dual process of $(\xi_t)_{t\geq 0}$ with respect to the duality function $H : S \times \tilde{S} \to \{0,1\}$

$$H(\xi, \tilde{\xi}) = \mathbb{1}\{\xi \cap \tilde{\xi} = \varnothing\},$$

where $\tilde{S} \subset S$ is the set of configurations with finitely many infected sites. Denoting by $\overline{\nu}$ the upper invariant law of $(\xi_t)_{t\geq 0}$ show that

$$\overline{\nu}\{\eta \in S : o \in \eta\} = \lim_{t\to\infty} \mathbb{P}(\tilde{\xi}_t^o \neq \varnothing).$$

(Note that you do not need to find the dual process to solve the exercise.)

Exercise 5 (15 pts)
Prove that for any $N \in \mathbb{Z}^+$

$$\mathbb{P}(\xi_t^o \subseteq [-N,N]^2 \ \forall t \geq 0 \mid \tau = \infty) = 0.$$